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DYNAMIC PROPERTIES OF RESILIENT MATERIALS: CONSTITUTIVE EQUATIONS

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CONTENTS

Constitutive equations are formulated for a class of resilient materials for which the stress distribution at any instant is assumed to depend both upon the deformation and upon the time rates of variation of the tensors defining it. Particular attention is given to aeolotropic bodies, the stress deformation relations for orthotropic and transversely isotropic materials being put in forms which exhibit the symmetry properties of the material. In the discussion of symmetry properties, attention is confined to the case where the stress tensor is a polynomial function of two only of the kinematic tensors.

Convected co-ordinate systems are employed in the development of the theory, but the method of transformation of the equations to a fixed frame of reference is also given. The modifications which are required for materials exhibiting curvilinear aeolotropy are briefly indicated, and some discussion is included of the manner in which certain types of geometrical constraint can be accounted for in the stress-deformation relations.

1. Introduction

Much recent work in continuum mechanics has been concentrated on the derivation of constitutive equations in closed form for materials, for which the mechanical properties may be described by means of relations between the quantities which describe the state of stress and the deformation and their time rates of variation. For example, Truesdell (1955 *a*, *b*, *c*) has developed relations for hypo-elastic materials by starting from the concept of rate of stress as a function of stress and rate of deformation and restricting the form of his equations by dimensional analysis. The connexion of this theory with the theory of finite elastic deformation has been examined by Noll (1955) and its relation to some of the classical theories of plasticity has been discussed by Green (1956 *a*, *b*). Rivlin & Ericksen (1955) on the other hand, have considered materials for which the stress may be considered as a general function of the space derivatives of the displacement, velocity and successive accelerations, and have shown that these assumptions lead to a representation of the stress in terms of a series of kinematic tensors. By making use of the properties of

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symmetric matrices, the constitutive equations for isotropic bodies may then be reduced to closed form. This procedure, however, leaves possible indeterminacies in the representation, a difficulty which Rivlin avoids in subsequent work (1955, 1956) by assuming the stress to be expressible as a polynomial in two only of the kinematic tensors.

For problems involving plastic flow or the motion of fluids, it is usually the situation at a point of space which is of interest. A fixed Euclidean frame of reference is then employed, and when time derivatives of the stress or deformation enter into the constitutive equations these must be formed in a manner which gives the right invariance properties. The laws of formation of such time derivatives have been given in general form by Oldroyd (1950) and special cases have been discussed from several points of view by Truesdell (1955 *b*), Rivlin & Ericksen (1955) and by Cotter & Rivlin (1955). Alternatively, the situation in a given element of the material throughout the motion may be considered, by referring all equations to a convected co-ordinate system which is associated with points of the body and moves with it as it is deformed. This method has also been discussed by Oldroyd, and has been used in the solution of problems by Lodge (1951).

In the work of Rivlin & Ericksen (1955) and of Rivlin (1955) attention has been concentrated on isotropic bodies. When the strain tensor appears explicitly in the constitutive equations there is a preferred initial configuration and aeolotropic properties may arise. Materials of this type are considered in the present paper, it being assumed that the stresss depends both upon the deformation and upon the time rates of variation of the quantities defining it. Such materials are here described as 'resilient' to distinguish them from the bodies considered in the static theory of finite elastic deformation in which rate of strain effects are ignored in the formulation of mechanical properties. Physically, the equations might be expected to give a description of dynamic hysteresis, a phenomenon which occurs in rubber-like materials.

It is therefore assumed that the mechanical properties of a homogeneous (rectilinearly) aeolotropic body may be described by means of an expression for the stress tensor, in terms of kinematic tensors which specify the deformation and its successive time rates of variation, these tensors being referred to a convected co-ordinate system which coincides, in the undeformed body at rest at time $t = 0$, with a fixed rectangular Cartesian frame of reference. As in the theory of elasticity, the general equations may be modified to indicate the existence of symmetries in the material. An exhaustive study of the forms appropriate to a general range of crystal classes could probably be made by making use of classical invariant theory (see, for example, Turnbull 1928; Weyl 1939). In the present paper an independent treatment is given of two cases of common interest, orthotropy and transverse isotropy. To avoid possible indeterminacies in the representation it is assumed that the mechanical properties can be represented by an expression for the stress tensor, as a polynomial in the tensors defining the deformation and its first time rate of variation. The equations for the orthotropic case follow directly from a consideration of the products of the kinematic tensors, which remain invariant in form under the required transformation of co-ordinates. For transversely isotropic bodies, a reduction of the equations to closed form may be achieved by making use of the Hamilton–Cayley theorem and the generalizations of this theorem for 3×3 matrices given by Rivlin (1955). As a result of this reduction, the elements of the stress tensor are expressed as polynomials in the elements of the kinematic tensors. The relations for transversely isotropic bodies lead naturally to a formula for the isotropic case, and the relation of this result to other forms is discussed.

An indication is given of the modifications required for curvilinearly aeolotropic bodies; the influence of certain types of geometrical constraint upon the form of the constitutive equations is also examined briefly. Finally, a summary is given of the method of transformation of the equations to a fixed frame of reference.

2. Notation and formulae

The approach of the present paper using convected co-ordinate systems corresponds in many respects to that employed by Oldroyd (1950), Green (1956*a*) and other workers, but some detailed changes in notation are necessary for convenience in the subsequent analysis.

A system of convected curvilinear co-ordinates θ^i is associated with elements of the material and moves with the body as it is deformed. The covariant and contravariant metric tensors for this system at an initial time $t = 0$ are $\gamma_{ij} = \gamma_{ij}(\theta^r, 0)$ and $\gamma^{ij} = \gamma^{ij}(\theta^r, 0)$ and at time t these become $\Gamma_{ij} = \Gamma_{ij}(\theta^r, t)$, $\Gamma^{ij} = \Gamma^{ij}(\theta^r, t)$, respectively. If $\mathrm{d}s_0$, $\mathrm{d}s$ are corresponding elements of length at time $t = 0$ and at the current time t, respectively, we derive from a consideration of the expression

$$\mathrm{d}s^2 - \mathrm{d}s_0^2 = (\Gamma_{ij} - \gamma_{ij})\,\mathrm{d}\theta^i\,\mathrm{d}\theta^j, \tag{2·1}$$

the definition

$$\eta_{ij} = \tfrac{1}{2}(\Gamma_{ij} - \gamma_{ij}) \tag{2·2}$$

for the covariant strain tensor η_{ij}, and by differentiation the formula

$$\alpha_{ij}^{(1)} = \alpha_{ij} = \frac{\mathrm{D}\eta_{ij}}{\mathrm{D}t} = \frac{1}{2}\frac{\mathrm{D}\Gamma_{ij}}{\mathrm{D}t}, \tag{2·3}$$

for the covariant rate of strain tensor α_{ij}, $\mathrm{D}/\mathrm{D}t$ denoting differentiation with respect to t holding the convected co-ordinates θ^i constant. Higher rate of strain tensors $\alpha_{ij}^{(r)}$ may be obtained if required by successive differentiation. Thus

$$\alpha_{ij}^{(r)} = \frac{\mathrm{D}^{r-1}\alpha_{ij}}{\mathrm{D}t^{r-1}} = \frac{\mathrm{D}^r\eta_{ij}}{\mathrm{D}t^r} = \frac{1}{2}\frac{\mathrm{D}^r\Gamma_{ij}}{\mathrm{D}t^r} \quad (r = 2, 3, 4, \ldots). \tag{2·4}$$

Mixed and contravariant kinematic tensors may be formed in several ways according to the choice of metric. In the present paper we shall employ the metric tensors γ^{ij}, γ_{ij} to raise and lower affixes, and write

$$\eta^i_j = \gamma^{ik}\eta_{kj}, \quad \eta^{ij} = \gamma^{ik}\eta^j_k = \gamma^{ik}\gamma^{jl}\eta_{kl},$$

with corresponding expressions for the remaining kinematic tensors.

We denote by τ^{ij} the contravariant stress tensor, by f^i, F^i the acceleration and body force vectors, respectively, and by ρ the density, all referred to the convected reference frame θ^i at time t. The equations of motion then take the form

$$\tau^{ij}_{,j} + \rho F^i = \rho f^i, \tag{2·5}$$

the comma signifying covariant differentiation with respect to the variables θ^i and the metric tensors Γ_{ij} and Γ^{ij}.

Following the approach adopted by the writer (1956) for elastic systems, we introduce further convected reference frames ${}^{(r)}\theta^i$ $(r = 1, 2, 3, \ldots)$ for the specification of non-isotropic properties. With each of these systems may be associated kinematic and mechanical tensors ${}^{(r)}\gamma_{ij}$, ${}^{(r)}\Gamma_{ij}$, ${}^{(r)}\eta_{ij}$, ${}^{(r)}\alpha_{ij}^{(n)}$, ${}^{(r)}\tau^{ij}$, ${}^{(r)}F^i$, ${}^{(r)}f^i$, ..., which are strictly analogous to the corresponding unnumbered quantities for the co-ordinate system θ^i, and may be related to them in the usual manner by tensor transformations. For example

$${}^{(r)}\eta_{ij} = \frac{\partial\theta^l}{\partial{}^{(r)}\theta^i}\frac{\partial\theta^m}{\partial{}^{(r)}\theta^j}\eta_{lm}.$$

We denote by x^i a convected co-ordinate system which coincides at time $t = 0$ with a rectangular Cartesian reference frame X^i in the undeformed body; the stress, strain and rate of strain tensors which correspond to the quantities τ^{ij}, η_{ij}, α_{ij} in the system θ^i are then denoted by t^{ij}, e_{ij}, a_{ij}, respectively. With this choice of co-ordinates, the metric tensors γ_{ij}, γ^{ij} reduce to Kronecker deltas and the contravariant, mixed and covariant components of the kinematic tensors are equal; in any subsequent expressions involving these quantities, the most convenient of the equivalent forms

$$e_{ij} = e^i_j = e^j_i = e^{ij}, \quad a_{ij} = a^i_j = a^j_i = a^{ij},$$

will therefore be employed; the symbols **E**, **A** are used to represent the symmetric matrices with elements e_{ij}, a_{ij}, respectively.

3. Aeolotropic bodies

For materials which possess a preferred initial configuration, it is evidently possible to postulate aeolotropic properties and to relate these to the body at rest in that configuration. We therefore consider a material which, at an initial time $t = 0$ is unstressed and at rest, the density ρ_0 being uniform in this initial state. In any subsequent continuously varying deformation, the contravariant stress tensor t^{ij} is supposed to be expressible as a tensor function

$$t^{ij} = f^{ij}(e_{rs}, a_{pq}) \tag{3·1}$$

of the kinematic tensors e_{rs}, a_{pq}, the functions f^{ij} being symmetric polynomials in their arguments. The material thus defined is initially homogeneous in the sense that a translation of the origin of the X^i reference frame (which determines the convected system x^i) leaves the form of (3·1) unchanged; aeolotropic properties arise since the form of t^{ij} is, in general, altered by an arbitrary rotation of the X^i co-ordinate axes.

We may here observe that in the case of elastic materials, the stress strain relations for homogeneous aeolotropic bodies may be written

$$t^{ij} = \frac{1}{2\sqrt{I}}\left(\frac{\partial W}{\partial e_{ij}} + \frac{\partial W}{\partial e_{ji}}\right), \tag{3·2}$$

(see, for example, Green & Zerna 1954), where the strain energy function W is a function, apart from constants, only of the components e_{rs}, and where

$$I = |\,\delta_{rs} + 2e_{rs}\,|.$$

Except for the factor $I^{-\frac{1}{2}}$, which itself may be approximated by a polynomial expression in e_{rs}, the formula (3·2) may be regarded as a special case of (3·1), and in this sense, the latter definition may be considered to be a natural generalization of that commonly employed for elastic bodies.

Evidently any discussion of symmetry properties based on (3·1) may also be applied to any definition in which the stresses t^{ij} are expressed as polynomial functions, which include, in addition to the arguments e_{rs}, a_{pq}, non-vanishing invariant functions of these quantities, such as $I^{-\frac{1}{2}}$, which are not, in themselves, polynomials in the kinematic tensors. For any such expression may be written

$$t^{ij} = \sum_{r=1}^{n} \phi_r f^{ij}_{(r)},$$

where ϕ_r are non-vanishing invariant functions of the kinematic tensors, whilst $f^{ij}_{(r)}$ are again symmetric polynomial functions of these quantities.

4. Orthotropy

For materials of the type now being considered it is evidently possible to postulate symmetry properties analogous to those occurring in ideally elastic bodies. We may, for example, regard a material as being orthotropic if its mechanical properties at any point are symmetrical when referred to the planes $X^i = \text{constant}$ (or $x^i = \text{constant}$) in the undeformed body at rest at the initial time $t = 0$. This implies that the stress tensor t^{ij} when expressed as a polynomial function

$$t^{ij} = f^{ij}(e_{rs}, a_{pq}) \tag{4·1}$$

of the kinematic tensors e_{rs}, a_{pq} remains invariant in form under all transformations of the type

$$(X^1, X^2, X^3) = (\pm\overline{X}^1, \pm\overline{X}^2, \pm\overline{X}^3).\dagger \tag{4·2}$$

This condition may be expressed as

$$f^{ij}(e_{rs}, a_{pq}) = f^{ij}(\bar{e}_{rs}, \bar{a}_{pq}),$$

where $\bar{e}_{rs}$, $\bar{a}_{pq}$ are the transforms of e_{rs}, a_{pq} obtained by means of (4·2).

To determine the form of t^{ij} consider first the expression

$$\chi_{ij} = A_{rst....uvw} b_{ir} c_{st} \dots d_{uv} f_{wj}, \tag{4·3}$$

where $A_{rst....uvw}$ is a constant tensor and the symbols $b_{ij}, c_{ij} \dots d_{ij}, f_{ij} \dots$ are used, as required, to represent the corresponding component of one or other of the kinematic tensors e_{ij}, a_{ij} or else may be replaced by the Kronecker delta δ_{ij}. The quantity χ_{ij} transforms as a Cartesian tensor product with respect to rotations of the axes X^i. Under a transformation to the system $\overline{X}^i$, (4·3) yields

$$\begin{aligned}\bar{\chi}_{\lambda\mu} &= \frac{\partial X^i}{\partial \overline{X}^\lambda}\frac{\partial X^j}{\partial \overline{X}^\mu}\chi_{ij} \\ &= \frac{\partial X^i}{\partial \overline{X}^\lambda}\frac{\partial X^j}{\partial \overline{X}^\mu} A_{rst....uvw}\left(\frac{\partial \overline{X}^p}{\partial X^i}\frac{\partial \overline{X}^k}{\partial X^r}\bar{b}_{pk}\right)\left(\frac{\partial \overline{X}^l}{\partial X^s}\frac{\partial \overline{X}^m}{\partial X^t}\bar{c}_{lm}\right)\dots\left(\frac{\partial \overline{X}^n}{\partial X^u}\frac{\partial \overline{X}^p}{\partial X^v}\bar{d}_{np}\right)\left(\frac{\partial \overline{X}^q}{\partial X^w}\frac{\partial \overline{X}^\beta}{\partial X^j}\bar{f}_{q\beta}\right) \\ &= A_{rst....uvw}\left(\frac{\partial \overline{X}^k}{\partial X^r}\frac{\partial \overline{X}^l}{\partial X^s}\frac{\partial \overline{X}^m}{\partial X^t}\dots\frac{\partial \overline{X}^n}{\partial X^u}\frac{\partial \overline{X}^p}{\partial X^v}\frac{\partial \overline{X}^q}{\partial X^w}\right)\bar{b}_{\lambda k}\bar{c}_{lm}\dots\bar{d}_{np}\bar{f}_{q\mu},\end{aligned} \tag{4·4}$$

the latter form being obtained by a suitable rearrangement of factors. For any of the transformations (4·2), each of the derivatives $\partial\overline{X}^i/\partial X^k$ occurring in (4·4) reduces to $\pm\delta_{ik}$

† This evidently implies a corresponding transformation in the undeformed body of the convected system x^i; in this and the following section we use X^i to indicate that the initial state is being considered.

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the sign of this factor depending upon which of the transformations $X^k = \pm \overline{X}^k$ for the given value of k is chosen. In order that the form of χ_{ij} may remain unchanged, an even number of negative signs must occur whichever of the transformations (4·2) is selected. This requires that all components of the tensor $A_{rst\,....\,uvw}$ in which any given suffix occurs an odd number of times shall be zero. It is sufficient for the present purpose to assume that successive indices are restricted to have equal values in pairs and to write the product (4·3) as

$$\chi_{ij} = A_{rrtt\,..\,uuvv} b_{ir} c_{rt} \dots d_{uv} f_{vj}, \tag{4·5}$$

summation being carried out over all repeated indices. This assumption implies that the off-diagonal components $A_{kl\,........}$, $A_{..\,kl\,......}$, ..., $A_{........\,kl}$ are zero. The expression χ_{ij} may then be regarded as a sum of products

$$P_{ij} = b_{ir} c_{rs} d_{st} \dots l_{uv} m_{vj}, \tag{4·6}$$

with each pair of successive indices equal. In this equation and subsequently throughout the present section, unless otherwise indicated, the summation convention is no longer employed. Each product (4·6) is evidently unchanged in form under all transformations (4·2); in the sum χ_{ij} it is multiplied by the appropriate component of the tensor $A_{rrt\,....\,uvv}$. We observe that the assumption that successive indices are equal in the tensor $A_{rst\,....\,uvw}$ gives no loss of generality. Any other pairing of equal indices would merely give rise to a different order of factors in (4·6).

The product P_{ij} may be resolved into simpler factors by considering in detail the numerical values which its indices can assume. Consider the case where i and j are numerically different. For $r = i$ and $r = j$, we have respectively

$$\left.\begin{aligned} P_{ij} &= c_{is} d_{st} \dots l_{uv} m_{vj} (b_{ii}), \\ P_{ij} &= b_{ij} (c_{js} d_{st} \dots l_{uv} m_{vj}). \end{aligned}\right\} \tag{4·7}$$

If i, j and r are all numerically different, the next index s must be equal to one of them; the three possible cases yield

$$\left.\begin{aligned} P_{ij} &= d_{it} \dots l_{uv} m_{vj} (b_{ir} c_{ri}) \quad (s = i), \\ P_{ij} &= b_{ir} c_{rj} (d_{jt} \dots l_{uv} m_{vj}) \quad (s = j), \\ P_{ij} &= b_{ir} d_{rt} \dots l_{uv} m_{vj} (c_{rr}) \quad (s = r). \end{aligned}\right\} \tag{4·8}$$

In (4·7) and (4·8) each bracketed group of factors constitutes a product of the type

$$H = b_{kl} c_{lm} d_{mn} f_{np} \dots h_{qk}, \tag{4·9}$$

in which each index is repeated and which does not contain the indices i, j, whilst the remaining terms constitute a product of the same type as P_{ij} but with fewer factors. Moreover, this reduction is always possible provided P_{ij} contains three or more factors. By a repetition of the process it is therefore possible to reduce P_{ij} to the form

$$P_{ij} = R_{ij}\,\Theta, \tag{4·10}$$

in which R_{ij} contains either one or two factors

$$R_{ij} = b_{ij} \quad \text{or} \quad R_{ij} = b_{ir} c_{rj} \quad (i \neq r \neq j \neq i), \tag{4·11}$$

whilst Θ is a product of factors H of the type (4·9).

These latter factors may be further reduced to products of more elementary basic types by an extension of the foregoing process. Thus, if $k = l$, (4·9) may be written

$$H = (b_{kk}) \, . \, (c_{km} d_{mn} f_{np} \dots h_{qk}).$$

If $k \neq l$, but m is equal either to k or l we obtain the resolutions

$$H = (b_{kl} c_{lk}) \, . \, (d_{kn} f_{np} \dots h_{qk}) \quad (m = k),$$
$$H = (c_{ll}) \, . \, (b_{kl} d_{ln} f_{np} \dots h_{qk}) \quad (m = l).$$

If k, l and m are all different, n must be numerically equal to one of them, and the three possible cases then give

$$H = (b_{kl} c_{lm} d_{mk}) \, . \, (f_{kp} \dots h_{qk}) \quad (n = k),$$
$$H = (c_{lm} d_{ml}) \, . \, (b_{kl} f_{lp} \dots h_{qk}) \quad (n = l),$$
$$H = (d_{mm}) \, . \, (b_{kl} c_{lm} f_{mp} \dots h_{qk}) \quad (n = m).$$

Again, this reduction is always possible provided that H contains more than three factors; repetition of the process is sufficient to reduce H to a product of factors, each of which can be derived from

$$b_{ij} c_{jk} d_{ki} \quad (i \neq j \neq k \neq i) \tag{4·12}$$

by suitably giving to the quantities b_{ij}, c_{ij}, d_{ij} the values e_{ij}, a_{ij} or δ_{ij}.

A complete set of forms for the coefficients R_{ij} $(i \neq j)$ is obtained by replacing b_{ij}, c_{ij} in (4·11) in all possible ways by the corresponding components of the kinematic tensors **E**, **A** giving

$$\left.\begin{array}{l} e_{ij}, \quad a_{ij}, \\ e_{ir} e_{rj}, \quad a_{ir} a_{rj}, \quad e_{ir} a_{rj}, \quad a_{ir} e_{rj} \\ (i \neq r \neq j \neq i, \; r \text{ not summed}). \end{array}\right\} \tag{4·13}$$

Similarly, for the terms which can occur in the products H we derive from (4·12) the 21 quantities

$$\left.\begin{array}{lr} e_{ii}, \quad a_{ii}, & \text{(i)} \\ e_{ij} e_{ji}, \quad a_{ij} a_{ji}, \quad e_{ij} a_{ji}, & \text{(ii)} \\ e_{ij} e_{jk} a_{ki}, \quad e_{ij} a_{jk} a_{ki}, & \text{(iii)} \\ (i \neq j \neq k \neq i; \; i, j, k \text{ not summed}), & \end{array}\right\} \tag{4·14}$$

there being six terms each of types (i) and (iii) and nine of type (ii) obtained by giving i, j, k all possible combinations of the values 1, 2, 3 subject to the stated restrictions. In addition we have the products

$$e_{12} e_{23} e_{31} \quad \text{and} \quad a_{12} a_{23} a_{31},$$

which may, however, be expressed in terms of the scalar invariants $\operatorname{tr} \mathbf{E}^3$ and $\operatorname{tr} \mathbf{A}^3$, respectively, together with the simpler combinations (4·14). For

$$\operatorname{tr} \mathbf{E}^3 = \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} e_{ij} e_{jk} e_{ki}$$
$$= 6 e_{12} e_{23} e_{31} + e_{11}^3 + e_{22}^3 + e_{33}^3 + 3\{e_{11}(e_{12}^2 + e_{13}^2) + e_{22}(e_{12}^2 + e_{23}^2) + e_{33}(e_{13}^2 + e_{23}^2)\}. \tag{4·15}$$

The assumption that t^{ij} is a polynomial in the kinematic tensors implies that it may be expressed as a linear combination of terms

$$\chi_{ij} + \chi_{ji},$$

where χ_{ij} $(=\chi^{ij})$ is given by (4·3). Each of these terms may be reduced by the foregoing process to a sum of terms of the type (4·10) in which the coefficients R_{ij} take the forms (4·13). We note that for $i=j$ any product may be resolved into factors of the types (4·14), (4·15) alone. By collecting together terms with like coefficients R_{ij} we obtain the formulae

$$\left.\begin{aligned} t^{11} &= \Theta_1, \\ t^{12} &= e_{12}\Theta_2 + a_{12}\Theta_3 + e_{13}e_{32}\Theta_4 + a_{13}a_{32}\Theta_5 + e_{13}a_{32}\Theta_6 + e_{23}a_{31}\Theta_7, \end{aligned}\right\} \tag{4·16}$$

with similar expressions for the remaining components of stress. These may be written in tensor form as

$$\begin{aligned} t^{ij} &= \delta^i_t\delta^j_t\Theta_{tt} \\ &\quad + \epsilon_{rst}\epsilon_{rst}\delta^i_r\delta^j_s\{e_{rs}\Theta^{(1)}_{tt} + a_{rs}\Theta^{(2)}_{tt} + e_{rt}e_{ts}\Theta^{(3)}_{tt} + a_{rt}a_{ts}\Theta^{(4)}_{tt} + e_{rt}a_{ts}\Theta^{(5)}_{tt} + e_{st}a_{tr}\Theta^{(6)}_{tt}\}, \end{aligned} \tag{4·17}$$

where summation is carried out over all repeated indices and ϵ_{rst} is equal to $+1$ or -1 according as r, s, t is an even or odd permutation of 1, 2, 3 and is equal to 0 otherwise. The functions Θ_r, Θ_{tt}, $\Theta^{(r)}_{tt}$ which occur in (4·16), (4·17) are of the same type as Θ, and apart from constant tensors which define the properties of the material contain the kinematic tensors e_{rs}, a_{rs} only in the combinations (4·14), (4·15). To preserve the symmetry of the tensor t^{ij}, the forms of Θ_6, Θ_7 must be such that interchange of the suffixes 1 and 2 in the components of e_{rs}, a_{pq} occurring in them, must have the effect of transforming Θ_6 into Θ_7 and Θ_7 into Θ_6; similarly, the functions $\Theta^{(5)}_{tt}$, $\Theta^{(6)}_{tt}$ must be such that they transform into each other in the same way. In (4·17) the functions Θ_{tt}, $\Theta^{(r)}_{tt}$ $(r=1$ to $5)$ are written as Cartesian tensor functions of the kinematic tensors.

The components of stress τ^{ij} referred to any other system of convected co-ordinates θ^i are obtained in the usual manner by tensor transformations. This procedure yields

$$\begin{aligned} \tau^{ij} &= \frac{\partial\theta^i}{\partial x^r}\frac{\partial\theta^j}{\partial x^s}t^{rs} \\ &= A^{ij}_{tt}\Theta_{tt} + \epsilon_{rst}\epsilon_{rst}A^{ij}_{rs}\{e_{rs}\Theta^{(1)}_{tt} + a_{rs}\Theta^{(2)}_{tt} + e_{rt}e_{ts}\Theta^{(3)}_{tt} + a_{rt}a_{ts}\Theta^{(4)}_{tt} + e_{rt}a_{ts}\Theta^{(5)}_{tt} + e_{st}a_{tr}\Theta^{(6)}_{tt}\}, \end{aligned} \tag{4·18}$$

where
$$A^{ij}_{rs} = \frac{\partial\theta^i}{\partial x^r}\frac{\partial\theta^j}{\partial x^s}. \tag{4·19}$$

Equation (4·18) reduces to the result derived by Green & Wilkes (1954) for elastic materials if we omit all terms containing a_{ij} and write

$$\Theta_{tt} = \frac{1}{\sqrt{I}}\frac{\partial W}{\partial e_{tt}}, \quad \Theta^{(1)}_{tt} = \frac{1}{\sqrt{I}}\frac{\partial W}{\partial(e^2_{rs})} \quad (r\neq s\neq t\neq r),$$

$$\Theta^{(3)}_{11} = \Theta^{(3)}_{22} = \Theta^{(3)}_{33} = \frac{1}{2\sqrt{I}}\frac{\partial W}{\partial(e_{12}e_{23}e_{31})},$$

where the strain energy function W is a function only of the components e_{ij}.

5. Transversely isotropic materials

As in § 4 we employ a definition of transverse isotropy analogous to that used in the theory of elasticity. Thus a material is defined as being transversely isotropic with respect to the x^i-direction, if its mechanical properties, again referred to the x^i co-ordinate system in the undeformed body at rest, are symmetrical at each point about a line through that point parallel to the x^1-axis (or the X^1-axis with which it coincides), and about the plane normal to it. This implies that the polynomial expression (4·1) for t^{ij} is now form invariant under all transformations of co-ordinates X^i in the undeformed body of the type

$$X^1 = \pm \overline{X}^1, \quad X^\alpha = X^\alpha(\overline{X}^2, \overline{X}^3) \quad (\alpha = 2, 3), \tag{5·1}$$

where $\overline{X}^\alpha$ is any arbitrary curvilinear system in the X^2, X^3-plane.

Evidently, t^{ij} must be restricted in form to a sum of terms of the type (4·5), since all of the transformations (4·2) are particular cases of (5·1). In addition, form invariance under the latter, more general, transformation requires that the components of $A_{rrt\,....\,uvv}$ obtained by giving any one suffix the values 2, 3, respectively, must be equal. Thus

$$A_{22tt\,...\,uvv} = A_{33tt\,...\,uvv}, \quad A_{rr22\,...\,uvv} = A_{rr33\,...\,uvv}, \ldots, \quad A_{rrtt\,...\,u22} = A_{rrtt\,...\,u33}. \tag{5·2}$$

This result follows by considering the behaviour of χ_{ij} under a general transformation of the type (5·1). For, in the notation of § 4 we obtain from (4·5)

$$\overline{\chi}_{\lambda\mu} = A_{rrt\,....\,uvv}\left(\frac{\partial \overline{X}^k}{\partial X^r}\frac{\partial \overline{X}^l}{\partial X^r}\right)\left(\frac{\partial \overline{X}^m}{\partial X^t}\cdots\right)\cdots\left(\cdots\frac{\partial \overline{X}^n}{\partial X^u}\right)\left(\frac{\partial \overline{X}^p}{\partial X^v}\frac{\partial \overline{X}^q}{\partial X^v}\right)\overline{b}_{\lambda k}\overline{c}_{lm}\cdots\overline{d}_{np}\overline{f}_{q\mu}. \tag{5·3}$$

If now $\overline{X}^i$ is an orthogonal Cartesian system which is obtained from the reference frame X^i by a small arbitrary rotation about the X^1-axis, we have

$$\frac{\partial \overline{X}^r}{\partial X^s} = \delta_{rs} - \xi_{rs}, \tag{5·4}$$

where ξ_{rs} is an antisymmetric tensor given by

$$\xi_{rr} = \xi_{1r} = \xi_{r1} = 0, \quad \xi_{23} = -\xi_{32}, \tag{5·5}$$

the non-zero components being small compared with unity. Neglecting the second and higher orders of small quantities, from (5·3) and (5·4) we have

$$\begin{aligned}\overline{\chi}_{\lambda\mu} = A_{rrt\,....\,uvv}\overline{b}_{\lambda r}\overline{c}_{rt}\cdots\overline{d}_{uv}\overline{f}_{v\mu} - A_{rrt\,....\,uvv}\{&\xi_{kr}\overline{b}_{\lambda k}\overline{c}_{rt}\cdots\overline{d}_{uv}\overline{f}_{v\mu} + \xi_{lr}\overline{b}_{\lambda r}\overline{c}_{lt}\cdots\overline{d}_{uv}\overline{f}_{v\mu}\\ &+\xi_{mt}\overline{b}_{\lambda r}\overline{c}_{rm}\cdots\overline{d}_{uv}\overline{f}_{v\mu} + \ldots + \xi_{nu}\overline{b}_{\lambda r}\overline{c}_{rt}\cdots\overline{d}_{nv}\overline{f}_{v\mu}\\ &+\xi_{pv}\overline{b}_{\lambda r}\overline{c}_{rt}\cdots\overline{d}_{up}\overline{f}_{v\mu} + \xi_{qv}\overline{b}_{\lambda r}\overline{c}_{rt}\cdots\overline{d}_{uv}\overline{f}_{q\mu}\}.\end{aligned} \tag{5·6}$$

The second group of terms in this expression must evidently vanish if form invariance is to be preserved. This implies that the coefficient of each product $\overline{b}_{\lambda r}\overline{c}_{st}\cdots\overline{d}_{uv}\overline{f}_{w\mu}$ for any given set of values of the indices $r, s, t \ldots u, v, w$ which can occur in the second group of terms in (5·6) must vanish separately. Remembering (5·5) we see that the coefficients of all products

$$\overline{b}_{\lambda L}\overline{c}_{Mt}\cdots\overline{d}_{uv}\overline{f}_{v\mu} \quad (t \ldots u, v \text{ not summed}),$$

in which $t \ldots u, v$ can be given any of the values 1, 2 or 3, whilst L, M take the pairs of values (1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (1, 3) and (3, 1), vanish identically. The remaining possible pairs of values are (2, 3) and (3, 2), for which we obtain the terms

$$(A_{33t\,....\,uvv}\xi_{23} + A_{22t\,....\,uvv}\xi_{32})\,\overline{b}_{\lambda 2}\overline{c}_{3t}\cdots\overline{d}_{uv}\overline{f}_{v\mu},$$

and

$$(A_{22t\,....\,uvv}\xi_{32} + A_{33t\,....\,uvv}\xi_{23})\,\overline{b}_{\lambda 3}\overline{c}_{2t}\cdots\overline{d}_{uv}\overline{f}_{v\mu},$$

respectively. For these expressions to vanish evidently requires that $A_{22t\,....\,uvv} = A_{33t\,....\,uvv}$. A similar examination of the products which are obtained by giving successively to each pair of adjacent suffixes belonging to different factors of the expression $\bar{b}_{\lambda r}\bar{c}_{st}\ldots\bar{d}_{uv}\bar{f}_{w\mu}$ all possible values, yields the remainder of the conditions (5·2).

These conditions have been derived by assuming the tensors b_{ij}, c_{ij}, ... to be distinct, so that any given set of numerical values of $r, t, \ldots, u, v$ determines uniquely each product of the set $b_{ir}c_{rt}\ldots d_{uv}f_{vj}$ obtained by giving i, j the values 1, 2, 3. Since, however, each of the symbols b_{ij}, c_{ij}, ... can represent one of only two quantities, e_{ij} and a_{ij}, it would appear feasible that in some cases two or more identical products could be obtained by suitable permutations of the indices $r, t, \ldots, u, v$. The procedure of picking out coefficients of products $\bar{b}_{\lambda r}\bar{c}_{st}\ldots\bar{d}_{uv}\bar{f}_{wj}$ in (5·6) with given numerical values of the indices $r, s, t, \ldots, u, v, w$ would then yield, in place of (5·2), a set of relations, in which several of the conditions (5·2) are replaced by a smaller number of equations which are, however, linear combinations of the relations derived by assuming the tensors b_{ij}, c_{ij}, ... to be distinct. Conditions relating individual components of the tensor $A_{rrt\ldots\ldots uvv}$ are then replaced by relations between linear combinations of these components. An examination of a general expression of the type (5·6) shows that the possibility of this type of non-uniqueness occurring does not affect the final result; when interchangeability of the indices is taken into account, the components of the tensor $A_{rrt\ldots\ldots uvv}$ which occur grouped together in the conditions imposed by the symmetry restrictions (5·1), occur also in the same combinations in the expansion of χ_{ij}. Suppose, for example, that the structure of the product

$$A_{p\,..\,qrs\,...\,tuv\,..\,w}b_{ip}\ldots c_{qr}d_{rs}\ldots f_{tu}h_{uv}\ldots l_{wj},\dagger$$

considered as a function of e_{rs}, a_{pq} can be such that the indices r and u are interchangeable, so that

$$b_{ip}\ldots c_{q\alpha}d_{\beta s}\ldots f_{t\lambda}h_{\mu v}\ldots l_{wj} = b_{ip}\ldots c_{q\lambda}d_{\mu s}\ldots f_{t\alpha}h_{\beta v}\ldots l_{wj},$$

for all values of the suffixes $i, p, \ldots, q, \alpha, \beta, s, \ldots, t, \lambda, \mu, v, \ldots, w, j$. The procedure employed in deriving the conditions (5·2) then yields

$$\begin{aligned} 2A_{p\,..\,q2s\,...\,t2v\,..\,w} &= A_{p\,..\,q2s\,...\,t3v\,..\,w} + A_{p\,..\,q3s\,...\,t2v\,..\,w} \\ &= 2A_{p\,..\,q3s\,...\,t3v\,..\,w}, \end{aligned}$$

in place of the conditions

$$\begin{aligned} A_{p\,..\,q2s\,...\,t2v\,..\,w} &= A_{p\,..\,q2s\,...\,t3v\,..\,w} \\ &= A_{p\,..\,q3s\,...\,t3v\,..\,w} = A_{p\,..\,q3s\,...\,t2v\,..\,w}, \end{aligned}$$

which would have been obtained by regarding the tensors $b_{ip}, c_{qr}, \ldots, l_{wj}$ as distinct. In either case the group of terms containing these coefficients may be reduced to

$$A_{p\,..\,q2s\,...\,t2v\,..\,w}\{b_{ip}\ldots c_{q2}d_{2s}\ldots f_{t2}h_{2v}\ldots l_{wj} + 2b_{ip}\ldots c_{q3}d_{3s}\ldots f_{t2}h_{2v}\ldots l_{wj} + b_{ip}\ldots c_{q3}d_{3s}\ldots f_{t3}h_{3v}\ldots l_{wj}\}.$$

In subsequent discussions we may therefore assume the conditions (5·2) to be satisfied uniquely.

Considering again the first pair of indices of the tensor $A_{rrt\ldots\ldots uvv}$ we see that two cases may arise:

$$A_{11tt\ldots uvv} = A_{22tt\ldots uvv} = A_{33tt\ldots uvv}, \qquad \text{(i)}$$

$$A_{11tt\ldots uvv} \neq A_{22tt\ldots uvv}. \qquad \text{(ii)}$$

† In this expression the repeated indices of the tensor $A_{pp\ldots\ldots ww}$ are written only once for brevity; thus

$$A_{p\,..\,qrs\,...\,tuv\,..\,w} = A_{pp\,....\,qqrrss\,......\,ttuuvv\,....\,ww}.$$

In the former case we may contract the first pair of indices and write

$$\chi_{ij} = B_{tt\ldots uvv} b_{ir} c_{rt} \ldots d_{uv} f_{vj}, \tag{5·7}$$

where $B_{tt\ldots uvv}$ is equal to each of the components (i). In the latter instance we may write the product as

$$\chi_{ij} = (A_{11tt\ldots uvv} - B_{tt\ldots uvv})\, b_{i1} c_{1t} \ldots d_{uv} f_{vj} + B_{tt\ldots uvv} b_{ir} c_{rt} \ldots d_{uv} f_{vj},$$

where $$B_{tt\ldots uvv} = A_{22tt\ldots uvv} = A_{33tt\ldots uvv},$$

the last term being of the form (5·7). From a similar consideration of the other indices we may infer that χ_{ij} can be regarded as a sum of products

$$P_{ij} = b_{ir} c_{rs} \ldots d_{uv} f_{vj}, \tag{5·8}$$

with summation over repeated indices, together with terms

$$(b_{ik} c_{kr} \ldots d_{s1}) \,.\, (f_{1t} \ldots h_{u1}) \ldots .. (l_{1v} \ldots m_{w1}), \tag{5·9}$$

in which one or more pairs of indices take the value unity. If we allow P_{ij}, Q_{ij}, $R_{ij} \ldots S_{ij}$ to represent products of the type (5·8), we see that (5·9) may be written as

$$P_{i1} Q_{1j} \,.\, R_{11} \ldots S_{11}, \tag{5·10}$$

after all possible invariant factors $R_{11} \ldots S_{11}$ have been extracted. The foregoing analysis is evidently unaffected if χ_{ij} is multiplied by any scalar polynomial function of the kinematic tensors which is form invariant under all transformations of the type (5·1). Corresponding to (5·8) and (5·10), we therefore obtain in the expression for χ_{ij} terms of the form

$$P_{ij}\Theta \quad \text{and} \quad P_{i1} Q_{1j} \Theta, \tag{5·11}$$

respectively, where in each case Θ represents a polynomial function of the kinematic tensors which is form invariant under the class of transformations (5·1). Remembering the symmetry property $t^{ij} = t^{ji}$ we may infer that t^{ij} may be expressed as a sum in which the two kinds of terms

$$\tfrac{1}{2}(P_{ij} + P_{ji})\,\Theta \quad \text{and} \quad \tfrac{1}{2}(P_{i1} Q_{1j} + P_{j1} Q_{1i})\,\Theta \tag{5·12}$$

occur. Under a tensor transformation

$$\tau^{ij} = \frac{\partial \theta^i}{\partial x^r} \frac{\partial \theta^j}{\partial x^s} t^{rs}$$

to the convected co-ordinate system θ^i these yield the expressions

$$\tfrac{1}{2}(\Pi^{ij} + \Pi^{ji})\,\Theta, \tag{5·13}$$

$$C^{ij}_{rs} P_{r1} Q_{1s} \Theta, \tag{5·14}$$

where $$C^{ij}_{rs} = \tfrac{1}{2}(A^{ij}_{rs} + A^{ij}_{sr}) = \frac{1}{2}\left(\frac{\partial \theta^i}{\partial x^r}\frac{\partial \theta^j}{\partial x^s} + \frac{\partial \theta^i}{\partial x^r}\frac{\partial \theta^j}{\partial x^s}\right),$$

and Π^{ij} is a product of the kinematic tensors η_{ij}, α_{ij} referred to the system θ^i, this product being obtained by transformation of the corresponding form (5·8) from the system X^i.

Further reduction of the expressions (5·13), (5·14) may be achieved by using the result that any symmetric polynomial $\mathbf{Q}$ in the two symmetric 3×3 matrices $\mathbf{E}$ and $\mathbf{A}$ may be expressed in the form

$$\mathbf{Q} = \phi_1 \mathbf{I} + \phi_2 \mathbf{E} + \phi_3 \mathbf{A} + \phi_4 \mathbf{E}^2 + \phi_5 \mathbf{A}^2 + \phi_6(\mathbf{EA} + \mathbf{AE}) + \phi_7(\mathbf{E}^2\mathbf{A} + \mathbf{AE}^2) + \phi_8(\mathbf{EA}^2 + \mathbf{A}^2\mathbf{E}) + \phi_9(\mathbf{E}^2\mathbf{A}^2 + \mathbf{A}^2\mathbf{E}^2), \tag{5·15}$$

where **I** is the unit matrix and ϕ_r are scalar functions containing the elements of **E** and **A** only as their scalar invariants. This result has been obtained by Rivlin (1955) by successive applications of the Hamilton–Cayley theorem, and from the method of proof it follows that the functions ϕ_r are necessarily polynomials in the elements of **E** and **A**. Moreover, since each of these functions ϕ_r may be absorbed into Θ without altering its character, the expansion (5·15) gives an immediate reduction of (5·13), and makes it possible to infer that the coefficients $\Pi^{ij}+\Pi^{ji}$ may be limited to the types

$$\left.\begin{array}{c}\gamma^{ij},\quad \eta^{ij},\quad \alpha^{ij},\quad \eta^{ik}\eta^j_k,\quad \alpha^{ik}\alpha^j_k,\quad \eta^{ik}\alpha^j_k+\alpha^{ik}\eta^j_k,\quad \eta^{ik}\eta^l_k\alpha^j_l+\alpha^{ik}\eta^l_k\eta^j_l,\\ \eta^{ik}\alpha^l_k\alpha^j_l+\alpha^{ik}\alpha^l_k\eta^j_l,\quad \eta^{ik}\eta^l_k\alpha^m_l\alpha^j_m+\alpha^{ik}\alpha^l_k\eta^m_l\eta^j_m.\end{array}\right\} \tag{5·16}$$

The factors Θ which occur in (5·13), (5·14) are polynomial functions of the scalar invariants

$$P_{ii}=b_{ij}c_{jk}\dots d_{li}, \tag{5·17}$$

of the kinematic tensors, and of the functions

$$P_{11}=b_{1j}c_{jk}\dots d_{l1}, \tag{5·18}$$

which are form invariant under the class of transformations (5·1). The number of independent forms (5·17) which can occur may be limited by using the result derived by Rivlin (1955), that any polynomial scalar invariant function of the kinematic matrices **E** and **A** may be expressed as a polynomial in the scalar invariants

$$\operatorname{tr}\mathbf{E},\quad \operatorname{tr}\mathbf{A},\quad \operatorname{tr}\mathbf{E}^2,\quad \operatorname{tr}\mathbf{A}^2,\quad \operatorname{tr}\mathbf{E}^3,\quad \operatorname{tr}\mathbf{A}^3,\quad \operatorname{tr}\mathbf{EA},\quad \operatorname{tr}\mathbf{E}^2\mathbf{A},\quad \operatorname{tr}\mathbf{EA}^2,\quad \operatorname{tr}\mathbf{E}^2\mathbf{A}^2, \tag{5·19}$$

where $$\operatorname{tr}\mathbf{E}=\operatorname{trace}\mathbf{E}=\eta^i_i=e_{ii},\quad \text{etc.} \tag{5·20}$$

Furthermore, we observe that any product

$$P_{11}=b_{1r}c_{rs}\dots d_{tu}f_{u1}$$

in the symmetric 3×3 matrices $b_{ij}, c_{ij}\dots f_{ij}$ may be regarded as the leading term in the elements of the symmetric 3×3 matrix **P** of which the general term is

$$P_{\lambda\mu}=\tfrac{1}{2}(b_{\lambda r}c_{rs}\dots d_{tu}f_{u\mu}+b_{\mu r}c_{rs}\dots d_{tu}f_{u\lambda}),$$

and can therefore be reduced by means of (5·15). This reduction gives an expression for P_{11} as a linear form in the quantities

$$\left.\begin{array}{c}e_{11},\quad a_{11},\quad e_{1r}e_{r1},\quad a_{1r}a_{r1},\quad e_{1r}a_{r1},\\ e_{1r}e_{rs}a_{s1},\quad e_{1r}a_{rs}a_{s1},\quad e_{1r}e_{rs}a_{st}a_{t1},\end{array}\right\} \tag{5·21}$$

with coefficients which are scalar polynomial functions of the invariants formed from e_{ij}, a_{ij}, or equivalently, from η^i_j, α^i_j. The functions Θ in (5·13) and (5·14) may therefore be expressed as polynomials in the combinations (5·19), (5·21) of the kinematic tensors.

To derive a manageable expression for τ^{ij} it is necessary to consider how many independent terms of the type (5·14) can be obtained. This involves the decomposition of the coefficients $A^{ij}_{rs}P_{r1}Q_{1s}$ into simpler forms. We may first, however, observe that the replacement of Q_{1s}, P_{r1} successively by Kronecker deltas yields the expressions

$$C^{ij}_{r1}P_{r1}\Theta,\quad C^{ij}_{1s}Q_{1s}\Theta,\quad C\,\Theta^{ji}_{11}, \tag{5·22}$$

which may be regarded as particular cases of (5·14).

6. Reduction of the equations for transversely isotropic bodies

A further simplification of the equations for the transversely isotropic case may be achieved by making use of the Hamilton–Cayley theorem

$$\mathbf{A}^3-\mathbf{A}^2\operatorname{tr}\mathbf{A}+\tfrac{1}{2}\mathbf{A}\{(\operatorname{tr}\mathbf{A})^2-\operatorname{tr}\mathbf{A}^2\}-\tfrac{1}{6}\mathbf{I}\{(\operatorname{tr}\mathbf{A})^3-3\operatorname{tr}\mathbf{A}\operatorname{tr}\mathbf{A}^2+2\operatorname{tr}\mathbf{A}^3\}=0, \qquad (6\cdot1)$$

for a 3×3 matrix $\mathbf{A}$, the generalization

$$\begin{aligned}&\mathbf{AEC}+\mathbf{EAC}+\mathbf{ACE}+\mathbf{ECA}+\mathbf{CAE}+\mathbf{CEA}\\&\quad=(\mathbf{EC}+\mathbf{CE})\operatorname{tr}\mathbf{A}+(\mathbf{CA}+\mathbf{AC})\operatorname{tr}\mathbf{E}+(\mathbf{AE}+\mathbf{EA})\operatorname{tr}\mathbf{C}\\&\qquad+\mathbf{A}(\operatorname{tr}\mathbf{EC}-\operatorname{tr}\mathbf{E}\operatorname{tr}\mathbf{C})+\mathbf{E}(\operatorname{tr}\mathbf{CA}-\operatorname{tr}\mathbf{C}\operatorname{tr}\mathbf{A})+\mathbf{C}(\operatorname{tr}\mathbf{AE}-\operatorname{tr}\mathbf{A}\operatorname{tr}\mathbf{E})\\&\qquad+\mathbf{I}(\operatorname{tr}\mathbf{A}\operatorname{tr}\mathbf{E}\operatorname{tr}\mathbf{C}-\operatorname{tr}\mathbf{A}\operatorname{tr}\mathbf{EC}-\operatorname{tr}\mathbf{E}\operatorname{tr}\mathbf{CA}-\operatorname{tr}\mathbf{C}\operatorname{tr}\mathbf{AE}+\operatorname{tr}\mathbf{AEC}+\operatorname{tr}\mathbf{CEA}),\end{aligned} \qquad (6\cdot2)$$

of this theorem obtained by Rivlin for three such matrices $\mathbf{A}$, $\mathbf{E}$, $\mathbf{C}$ and the formula

$$\begin{aligned}\mathbf{AEA}&=-\mathbf{A}^2\mathbf{E}-\mathbf{EA}^2+(\mathbf{AE}+\mathbf{EA})\operatorname{tr}\mathbf{A}+\mathbf{A}^2\operatorname{tr}\mathbf{E}+\mathbf{A}\{\operatorname{tr}\mathbf{AE}-\operatorname{tr}\mathbf{A}\operatorname{tr}\mathbf{E}\}\\&\quad+\tfrac{1}{2}\mathbf{E}\{\operatorname{tr}\mathbf{A}^2-(\operatorname{tr}\mathbf{A})^2\}+\mathbf{I}\{\operatorname{tr}\mathbf{A}^2\mathbf{E}-\operatorname{tr}\mathbf{A}\operatorname{tr}\mathbf{AE}-\tfrac{1}{2}\operatorname{tr}\mathbf{E}[\operatorname{tr}\mathbf{A}^2-(\operatorname{tr}\mathbf{A})^2]\},\end{aligned} \qquad (6\cdot3)$$

derived by putting $\mathbf{C}=\mathbf{A}$ in (6·2). In the subsequent analysis we shall denote by $\mathbf{P}(r)$ any matrix polynomial, none of whose terms is of power greater than r in the products of the matrices $\mathbf{A}$, $\mathbf{E}$ and $\mathbf{C}$, where the exact form of this polynomial is unimportant.† With this convention, (6·1) to (6·3) may be rewritten

$$\mathbf{A}^3=\mathbf{P}(2), \qquad (6\cdot1')$$

$$\mathbf{AEC}+\mathbf{EAC}+\mathbf{ACE}+\mathbf{ECA}+\mathbf{CAE}+\mathbf{CEA}=\mathbf{P}(2), \qquad (6\cdot2')$$

$$\mathbf{AEA}=-\mathbf{A}^2\mathbf{E}-\mathbf{EA}^2+\mathbf{P}(2), \qquad (6\cdot3')$$

respectively. From (6·3′) by replacing $\mathbf{A}$ by $\mathbf{A}^2$ and making use of (6·1′) we have

$$\mathbf{A}^2\mathbf{EA}^2=\mathbf{P}(4), \qquad (6\cdot4)$$

and similarly, by pre-multiplying (6·3′) by $\mathbf{A}$

$$\mathbf{A}^2\mathbf{EA}=-\mathbf{AEA}^2+\mathbf{P}(3). \qquad (6\cdot5)$$

We continue to allow $\mathbf{E}$ and $\mathbf{A}$ to denote the symmetric 3×3 matrices whose elements are e_{rs} and a_{rs}, respectively, and observe that in the expression for τ^{ij} the quantities $C^{ij}_{\lambda\mu}$ obtained by giving λ and μ the values 1, 2 and 3, also form a symmetric 3×3 matrix which we may therefore denote by $\mathbf{C}$. With this notation, the coefficients of Θ occurring in (5·14) and (5·22) may be written

$$\left.\begin{aligned}Q_{1s}C^{ij}_{sr}P_{r1}&=[\mathbf{QCP}]_{11},\\C^{ij}_{1r}P_{r1}=[\mathbf{CP}]_{11},\quad Q_{1s}C^{ij}_{s1}&=[\mathbf{QC}]_{11},\quad C^{ij}_{11}=[\mathbf{C}]_{11},\end{aligned}\right\} \qquad (6\cdot6)$$

where $\mathbf{P}$ and $\mathbf{Q}$ are arbitrary products of the matrices $\mathbf{A}$ and $\mathbf{E}$ of the types

$$\mathbf{P}=\mathbf{A}^{\alpha_1}\mathbf{E}^{\beta_1}\mathbf{A}^{\alpha_2}\mathbf{E}^{\beta_2}\ldots\mathbf{A}^{\alpha_n}\mathbf{E}^{\beta_n}\ldots, \qquad (6\cdot7)$$

and the symbol $[\ ]_{11}$ is employed to indicate that only the leading element P_{11} of the final product $\mathbf{P}$ is required. We may here observe that for any product $\mathbf{\Pi}$ of the symmetric matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, ..., $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$ we have the symmetry property

$$[\mathbf{\Pi}]_{11}=[\mathbf{ABC}\ldots\mathbf{XYZ}]_{11}=[\mathbf{ZYX}\ldots\mathbf{CBA}]_{11}. \qquad (6\cdot8)$$

† The coefficients in the polynomial $\mathbf{P}(r)$ are, in general, functions involving $\mathbf{A}$, $\mathbf{E}$ and $\mathbf{C}$ as their scalar invariants.

A preliminary simplification of the products (6·6) may be achieved by the method employed by Rivlin in deriving the formula (5·15). For example, any polynomial of the type (6·7) may be reduced to a sum of terms of similar type, but in which the indices α_r, β_r can only have the values 1 or 2, by making use of the Hamilton–Cayley theorem (6·1′). In each term of the resulting expression, terms of equal power in the same matrix may be combined by utilizing formulae of the type (6·3′) and (6·4), but in which $\mathbf{E}$ is replaced by some combination of the matrices $\mathbf{A}$, $\mathbf{E}$ and $\mathbf{C}$. By this means, the matrix products occurring in (6·6) may be reduced to a sum of terms, each of which consists of a product, in some order, of some or all of the five factors $\mathbf{E}$, $\mathbf{E}^2$, $\mathbf{A}$, $\mathbf{A}^2$ and $\mathbf{C}$, each factor occurring not more than once, and which in addition is multiplied by scalar invariants formed from the matrices $\mathbf{E}$, $\mathbf{A}$ and $\mathbf{C}$. We now consider the independent products of the matrices $\mathbf{E}$, $\mathbf{E}^2$, $\mathbf{A}$, $\mathbf{A}^2$ and $\mathbf{C}$ which may occur, starting with those of lowest power.

Remembering (6·8) we see that the coefficients of Θ of the first, second and third powers in $\mathbf{E}$, $\mathbf{A}$ and $\mathbf{C}$ which may occur may be taken to be

$$\left.\begin{array}{c}[\mathbf{C}]_{11}, \quad [\mathbf{CE}]_{11}, \quad [\mathbf{CA}]_{11}, \\ [\mathbf{CE}^2]_{11}, \quad [\mathbf{CA}^2]_{11},\end{array}\right\} \tag{6·9}$$

$$[\mathbf{CEA}]_{11}, \quad [\mathbf{CAE}]_{11}, \quad [\mathbf{ECA}]_{11}. \tag{6·10}$$

The last three are not independent, for, from (6·2′) and (6·8) we have the relation

$$[\mathbf{CEA}]_{11}+[\mathbf{CAE}]_{11}+[\mathbf{ECA}]_{11} = [\mathbf{P}(2)]_{11}, \tag{6·11}$$

so that in place of (6·10) we may take

$$[\mathbf{CEA}]_{11} \quad \text{and} \quad [\mathbf{CAE}]_{11} \tag{6·12}$$

as independent coefficients. By replacing $\mathbf{E}$ and $\mathbf{A}$ by $\mathbf{E}^2$ and $\mathbf{A}^2$, respectively, in (6·2′) and the derived relationship (6·11) we may infer that the products involving $\mathbf{C}$, $\mathbf{E}^2$, $\mathbf{A}$ and $\mathbf{C}$, $\mathbf{E}$, $\mathbf{A}^2$ may be limited to the types

$$[\mathbf{CE}^2\mathbf{A}]_{11}, \quad [\mathbf{CAE}^2]_{11}, \quad [\mathbf{CEA}^2]_{11}, \quad [\mathbf{CA}^2\mathbf{E}]_{11}. \tag{6·13}$$

For products involving the factors $\mathbf{C}$, $\mathbf{E}^2$, $\mathbf{A}^2$ we have, in addition to the relation

$$[\mathbf{CE}^2\mathbf{A}^2]_{11}+[\mathbf{CA}^2\mathbf{E}^2]_{11}+[\mathbf{E}^2\mathbf{CA}^2]_{11} = [\mathbf{P}(4)]_{11} \tag{6·14}$$

corresponding to (6·11), a further equation derived from alternative expressions for the product $\mathbf{EAECA}$. For reductions of the type (6·3′) yield

$$\begin{aligned}\mathbf{EAECA} &= (\mathbf{EAE})\mathbf{CA} = -\mathbf{E}^2\mathbf{ACA}-\mathbf{AE}^2\mathbf{CA}+\mathbf{P}(4)\\ &= \mathbf{E}^2\mathbf{A}^2\mathbf{C}+2\mathbf{E}^2\mathbf{CA}^2+\mathbf{A}^2\mathbf{E}^2\mathbf{C}+\mathbf{P}(4),\end{aligned}$$

and

$$\begin{aligned}\mathbf{EAECA} &= \mathbf{E}[\mathbf{A}(\mathbf{EC})\mathbf{A}] = -\mathbf{EA}^2\mathbf{EC}-\mathbf{E}^2\mathbf{CA}^2+\mathbf{P}(4)\\ &= \mathbf{A}^2\mathbf{E}^2\mathbf{C}+\mathbf{E}^2\mathbf{A}^2\mathbf{C}-\mathbf{E}^2\mathbf{CA}^2+\mathbf{P}(4),\end{aligned}$$

which imply that

$$\mathbf{E}^2\mathbf{CA}^2 = \mathbf{P}(4). \tag{6·15}$$

Remembering (6·14) we see that it is possible to choose either

$$[\mathbf{CE}^2\mathbf{A}^2]_{11} \quad \text{or} \quad [\mathbf{CA}^2\mathbf{E}^2]_{11}$$

as a single independent coefficient involving $\mathbf{C}$, $\mathbf{A}^2$ and $\mathbf{E}^2$.

Products of the factors **C**, **A**, $\mathbf{A}^2$ and **E** can evidently occur as

(i) $$[\mathbf{ACA}^2\mathbf{E}]_{11}, \quad [\mathbf{ACEA}^2]_{11}, \quad [\mathbf{CAEA}^2]_{11}, \tag{6·16}$$

(ii) the three products obtained from (i) by interchanging **A** and $\mathbf{A}^2$ and connected with them by formulae of the type (6·5), and

(iii) six further expressions obtained from (i) and (ii) by reversing the order of the factors, and equivalent to them by virtue of the symmetry property (6·8). Alternative expansions of the product **ACAEA** using (6·3′) and (6·5) now yield

$$(\mathbf{ACA})\mathbf{EA} = -\mathbf{A}^2\mathbf{CEA} - \mathbf{CA}^2\mathbf{EA} + \mathbf{P}(4) = \mathbf{ACEA}^2 + \mathbf{CAEA}^2 + \mathbf{P}(4),$$
$$\mathbf{A}(\mathbf{CAE})\mathbf{A} = -\mathbf{A}^2\mathbf{CAE} - \mathbf{CAEA}^2 + \mathbf{P}(4) = \mathbf{ACA}^2\mathbf{E} - \mathbf{CAEA}^2 + \mathbf{P}(4),$$
$$\mathbf{AC}(\mathbf{AEA}) = -\mathbf{ACEA}^2 - \mathbf{ACA}^2\mathbf{E} + \mathbf{P}(4),$$

and by addition of these three expressions we may infer that **ACAEA** can be expressed entirely in terms of products of the fourth and lower powers in **E**, **A** and **C**. It then follows that

$$\mathbf{ACEA}^2 = -\mathbf{CAEA}^2 + \mathbf{P}(4), \quad \mathbf{ACA}^2\mathbf{E} = \mathbf{CAEA}^2 + \mathbf{P}(4),$$

and therefore that all coefficients involving the factors **C**, **A**, $\mathbf{A}^2$ and **E** may be expressed in terms of

$$[\mathbf{CAEA}^2]_{11}, \tag{6·17}$$

and products involving lower powers of **C**, **A** and **E**. By a similar procedure, with **A** and **E** interchanged we obtain the further coefficient

$$[\mathbf{CEAE}^2]_{11}. \tag{6·18}$$

The extension (6·2′) of the Hamilton–Cayley theorem in the present instance yields no further independent relations between the fifth-power products.

Coefficients which may be formed from the factors **C**, **A**, $\mathbf{A}^2$ and $\mathbf{E}^2$ take the forms

$$[\mathbf{ACA}^2\mathbf{E}^2]_{11}, \quad [\mathbf{ACE}^2\mathbf{A}^2]_{11}, \quad [\mathbf{CAE}^2\mathbf{A}^2]_{11}, \tag{6·19}$$

corresponding to (6·16), together with expressions related to them either by relations of the type (6·5) or by the symmetry property (6·8). By successive application of (6·3′) to the product **AECAEA** we obtain

$$\begin{aligned}\mathbf{A}(\mathbf{EC})\mathbf{AEA} &= -\mathbf{A}^2\mathbf{ECEA} - \mathbf{ECA}^2\mathbf{EA} + \mathbf{P}(5)\\ &= \mathbf{A}^2\mathbf{E}^2\mathbf{CA} + \mathbf{A}^2\mathbf{CE}^2\mathbf{A} + \mathbf{CA}^2\mathbf{E}^2\mathbf{A} + \mathbf{P}(5)\\ &= \mathbf{A}^2\mathbf{E}^2\mathbf{CA} - \mathbf{ACE}^2\mathbf{A}^2 - \mathbf{CAE}^2\mathbf{A}^2 + \mathbf{P}(5)\\ &= \boldsymbol{\Phi}_1 \quad \text{(say)},\end{aligned}$$

and similarly

$$\mathbf{A}(\mathbf{ECAE})\mathbf{A} = \mathbf{A}^2\mathbf{E}^2\mathbf{CA} - \mathbf{ACA}^2\mathbf{E}^2 + \mathbf{CAE}^2\mathbf{A}^2 + \mathbf{P}(5) = \boldsymbol{\Phi}_2,$$
$$\mathbf{AEC}(\mathbf{AEA}) = -2\mathbf{A}^2\mathbf{E}^2\mathbf{CA} + \mathbf{ACA}^2\mathbf{E}^2 + \mathbf{ACE}^2\mathbf{A}^2 + \mathbf{P}(5) = \boldsymbol{\Phi}_3,$$
$$\mathbf{AE}(\mathbf{CA})\mathbf{EA} = \mathbf{A}^2\mathbf{E}^2\mathbf{CA} + \mathbf{ACA}^2\mathbf{E}^2 + \mathbf{ACE}^2\mathbf{A}^2 + \mathbf{P}(5) = \boldsymbol{\Phi}_4,$$

alternative expressions for $\boldsymbol{\Phi}_4$ becoming identical, as far as terms of the sixth power are concerned with $\boldsymbol{\Phi}_1$ and $\boldsymbol{\Phi}_2$. Since $\boldsymbol{\Phi}_3 = \boldsymbol{\Phi}_4$ it follows that $\mathbf{A}^2\mathbf{E}^2\mathbf{CA}$ can be expressed entirely in terms of products of the fifth and lower powers. A corresponding result follows for **AECAEA** from the expression for $\mathbf{\Phi}_1+\mathbf{\Phi}_2+\mathbf{\Phi}_3$. It is then seen without difficulty that

$$\begin{aligned}\mathbf{ACE^2A^2} &= -\mathbf{CAE^2A^2}+\mathbf{P}(5),\\ \mathbf{ACA^2E^2} &= \quad \mathbf{CAE^2A^2}+\mathbf{P}(5).\end{aligned}$$

A similar consideration of the product **AEACAE** shows that $\mathbf{ACA^2E^2}$ can be expressed entirely in terms of products of the fifth and lower powers, and yields the additional relations

$$\begin{aligned}\mathbf{E^2A^2CA} &= -\mathbf{A^2E^2CA}+\mathbf{P}(5) = \mathbf{P}(5),\\ \mathbf{A^2E^2AC} &= -\mathbf{A^2E^2CA}+\mathbf{P}(5) = \mathbf{P}(5).\end{aligned}$$

From these, and analogous results obtained from the expansions for **EACAEA** and **AEACEA** it follows that all products involving the factors **A**, $\mathbf{A}^2$, $\mathbf{E}^2$ and **C** may be expressed in terms of those of the fifth and lower powers in **E**, **A** and **C**. A similar result applies for products of the factors $\mathbf{A}^2$, **E**, $\mathbf{E}^2$ and **C**.

To reduce the products of all five factors **E**, $\mathbf{E}^2$, **A**, $\mathbf{A}^2$ and **C** we first observe that all such expressions either contain one of the factors **E** or **A** at the beginning or at the end, or else may be related to products of this type by making use of (6·5). Each of the combinations which need to be considered may therefore be regarded as a product of the factors **A**, $\mathbf{A}^2$, $\mathbf{E}^2$ and **C** premultiplied or post-multiplied by **E**, or as a product of the factors $\mathbf{A}^2$, **E**, $\mathbf{E}^2$ and **C** premultiplied or post-multiplied by **A**. Since each of these products of the sixth power in the matrices **A**, **E** and **C** can be expressed in terms of those of a lower power, without making use of the symmetry condition (6·8), it follows that each product of all five factors can be similarly reduced.

By successive application of the formulae (6·1) to (6·5) we have thus been able to reduce any coefficient of the type (6·6) to a sum of terms of two kinds. Each term of the first kind is a product of one of the quantities

$$\left.\begin{array}{lllll}[\mathbf{C}]_{11}, & [\mathbf{CE}]_{11}, & [\mathbf{CA}]_{11}, & [\mathbf{CE^2}]_{11}, & [\mathbf{CA^2}]_{11},\\ [\mathbf{CEA}]_{11}, & [\mathbf{CAE}]_{11}, & [\mathbf{CE^2A}]_{11}, & [\mathbf{CAE^2}]_{11}, & [\mathbf{CEA^2}]_{11},\\ [\mathbf{CA^2E}]_{11}, & [\mathbf{CE^2A^2}]_{11}, & [\mathbf{CA^2E^2}]_{11}, & [\mathbf{CEAE^2}]_{11}, & [\mathbf{CAEA^2}]_{11},\end{array}\right\} \qquad (6\cdot20)$$

with an expression which involves only the scalar invariants of **E** and **A** and may therefore be absorbed into the multiplying function Θ introduced in § 5. Each term of the second type consists of a scalar invariant product of the matrices **C**, **E** and **A**, involving **C** linearly, multiplied by a function of the matrices **E** and **A** which is invariant in form under transformations of the type (5·1).

This latter function may again be absorbed into Θ and since the factor involving **C** is a polynomial in the matrices **A**, **E**, **C** it may be written as a sum of terms of the type

$$\operatorname{tr}(\mathbf{PCQ}),$$

where **P**, **Q** are polynomials in the matrices **E** and **A**, together with coefficients which are scalar invariant functions of **E** and **A**. Remembering the definition of **C**, it follows that this second kind of term is equivalent to the form (5·13) which has already been considered, and from which there follows the set of coefficients (5·16).

The foregoing reductions thus lead to the conclusion that the coefficients of Θ which occur in (5·13), (5·14) can be expressed as linear combinations of the quantities (5·16), (6·20), respectively. Collecting these results into one formula, we obtain for the stress tensor

$$\begin{aligned}\tau^{ij} = {} & \gamma^{ij}\Theta_1+\eta^{ij}\Theta_2+\alpha^{ij}\Theta_3+\eta^{ik}\eta^j_k\Theta_4+\alpha^{ik}\alpha^j_k\Theta_5+(\eta^{ik}\alpha^j_k+\alpha^{ik}\eta^j_k)\,\Theta_6\\ & +(\eta^{ik}\eta^l_k\alpha^j_l+\alpha^{ik}\eta^l_k\eta^j_l)\,\Theta_7+(\eta^{ik}\alpha^l_k\alpha^j_l+\alpha^{ik}\alpha^l_k\eta^j_l)\Theta_8+(\eta^{ik}\eta^l_k\alpha^m_l\alpha^j_m+\alpha^{ik}\alpha^l_k\eta^m_l\eta^j_m)\,\Theta_9+C^{ij}_{11}\Psi_1\\ & +C^{ij}_{1r}\{e_{r1}\Psi_2+a_{r1}\Psi_3+e_{rs}e_{s1}\Psi_4+a_{rs}a_{s1}\Psi_5+e_{rs}a_{s1}\Psi_6+a_{rs}e_{s1}\Psi_7+e_{rs}e_{st}a_{t1}\Psi_8\\ & +a_{rs}e_{st}e_{t1}\Psi_9+e_{rs}a_{st}a_{t1}\Psi_{10}+a_{rs}a_{st}e_{t1}\Psi_{11}+e_{rs}e_{st}a_{tu}a_{u1}\Psi_{12}+a_{rs}a_{st}e_{tu}e_{u1}\Psi_{13}\\ & +e_{rs}a_{st}e_{tu}e_{u1}\Psi_{14}+a_{rs}e_{st}a_{tu}a_{u1}\Psi_{15}\},\end{aligned} \tag{6·21}$$

where Θ_r ($r = 1$ to 9) and Ψ_r ($r = 1$ to 15) are scalar invariant functions of the quantities (5·19), (5·21), these functions being polynomials in their arguments. We notice that this formula contains two groups of terms. In the first group, involving the functions Θ_r the coefficients are isotropic in character; in the second group, the coefficients are characteristic of the transversely isotropic case. By virtue of (6·14) and (6·15) one or other of the terms involving Ψ_{12}, Ψ_{13} may be omitted without loss of generality.

7. Isotropic bodies

If the second group of terms, involving the functions Ψ_r is omitted from (6·21) and the functions Θ_r in the remaining terms allowed to become functions of the invariants (5·19) alone, the resulting relationship defines an isotropic material. This result differs from the formulae employed by other workers (for example, Green 1956 *a*) in that the contravariant kinematic tensors are derived from the covariant forms by using the metric tensor γ^{ij} to raise indices in place of the more usual Γ^{ij}. The difference arises from the procedure employed in the present paper of referring the mechanical properties of the material to an initial configuration; in other work where plastic and fluid properties are being considered it is the current configuration which is usually of primary importance. The equations derived in this latter case must evidently be equivalent to the forms obtained from a consideration of isotropic properties in some initial configuration, and this equivalence is readily demonstrated.

Consider the invariants

$$\left.\begin{aligned} J_1 = \eta^r_r,\quad J_2 = \eta^r_s\eta^s_r,\quad J_3 = \eta^r_s\eta^s_t\eta^t_r,\\ I = |\,\delta^r_s+2\eta^r_s\,| = |\,\gamma^{ri}\Gamma_{is}\,| = |\,\Gamma_{ik}\,|/|\,\gamma_{ik}\,|.\end{aligned}\right\} \tag{7·1}$$

Between these, we may establish by a straightforward calculation the relation

$$\begin{aligned} 8J_3 &= 24\,|\,\eta^i_k\,|+12J_1J_2-4J_1^3\\ &= 3(I-1-2J_1-2J_1^2+2J_2)+12J_1J_2-4J_1^3. \end{aligned} \tag{7·2}$$

By differentiating this relation with respect to η_{ik} and substituting the results

$$\frac{\partial J_1}{\partial \eta_{ik}} = \gamma^{ik},\quad \frac{\partial J_2}{\partial \eta_{ik}} = 2\eta^{ik},\quad \frac{\partial J_3}{\partial \eta_{ik}} = 3\eta^k_r\eta^{ri},\quad \frac{\partial I}{\partial \eta_{ik}} = 2I\Gamma^{ik}, \tag{7·3}$$

we obtain

$$I\Gamma^{ik} = (1+2J_1-2J_2+2J_1^2)\,\gamma^{ik}-2(1+2J_1)\,\eta^{ik}+4\eta^k_r\eta^{ri}. \tag{7·4}$$

This is an expression for Γ^{ik} in terms of γ^{ik}, contravariant and mixed tensors derived from the covariant form by raising indices with γ^{ik} and invariants formed from these quantities.

We note further than I is equivalent to the non-vanishing quantity $(\rho_0/\rho)^2$. Any isotropic form in which the coefficients and invariants are formed from mixed and contravariant components of the kinematic tensors, which are derived from the covariant form by raising indices with the metric tensor Γ^{ik}, may thus be reduced to a corresponding form in which the initial metric tensor γ^{ik} performs this function. To complete the reduction, it may be necessary to simplify the coefficients of the latter form by making use of (5·15). The scalar invariant functions, corresponding to Θ_r in (6·21), which occur in the two forms are not, in general, identical.

8. Curvilinear aeolotropy

The preceding results may be generalized without difficulty to include curvilinearly aeolotropic bodies. In the rectilinear case, aeolotropic properties are defined by means of the convected system x^i which coincides at the initial time $t = 0$ with the Cartesian system X^i in the undeformed body. This system is replaced, for curvilinearly aeolotropic bodies by a convected system ${}^{(1)}\theta^i$ which coincides at time $t = 0$ with a suitably chosen curvilinear system $\bar{\theta}^i$ in the undeformed body at rest; for simplicity, we confine attention to the case where this latter system is orthogonal, so that

$$ {}^{(1)}\gamma^{ii} = 1/({}^{(1)}\gamma_{ii}), \quad {}^{(1)}\gamma_{ij} = {}^{(1)}\gamma^{ij} = 0 \quad (i \neq j). \tag{8·1}$$

In the case of ideally elastic bodies, curvilinearly aeolotropic properties are defined by specifying that the strain energy function W shall be a function, apart from physical constants independent of ${}^{(1)}\theta^i$ or t, only of the physical components of strain ${}^{(1)}\eta_{(ij)}$ defined by

$$ {}^{(1)}\eta_{(ij)} = \frac{{}^{(1)}\eta_{ij}}{\surd({}^{(1)}\gamma_{ii}{}^{(1)}\gamma_{jj})} = {}^{(1)}\eta_{(ji)}. \tag{8·2}$$

These latter components are introduced from the consideration that an element of length ds_i lying in the direction of a ${}^{(1)}\theta^i$ co-ordinate curve in the undeformed body is given by

$$ ds_i = \surd({}^{(1)}\gamma_{ii})\, d^{(1)}\theta^i \quad (i \text{ not summed}), \tag{8·3}$$

so that the formula (2·1) for $ds^2 - ds_0^2$ may be written

$$ ds^2 - ds_0^2 = 2^{(1)}\eta_{(ij)}\, ds_i\, ds_j. \tag{8·4}$$

It then follows without difficulty (Adkins 1955) that the quantities

$$ {}^{(1)}\tau_{(ij)} = {}^{(1)}\tau^{ij} \surd\{{}^{(1)}\gamma_{ii}{}^{(1)}\gamma_{jj}\} \quad (i, j \text{ not summed}) \tag{8·5}$$

are also functions, apart from physical constants, of the components $\eta_{(ij)}$.

A natural generalization of this result to materials in which the stress depends upon rates of deformation, is obtained by postulating that the components $\tau_{(ij)}$ shall be functions, apart from physical constants which are independent of θ^i and t, only of suitably defined physical components ${}^{(1)}\eta_{(ij)}$, ${}^{(1)}\alpha_{(ij)}$, ${}^{(1)}\alpha^{(p)}_{(ij)}$ of the kinematic tensors ${}^{(1)}\eta_{ij}$, ${}^{(1)}\alpha_{ij}$, ${}^{(1)}\alpha^{(p)}_{ij}$ referred to the co-ordinate system ${}^{(1)}\theta^i$ in the undeformed body at rest at time $t = 0$. The required physical components ${}^{(1)}\alpha_{(ij)}$, ${}^{(1)}\alpha^{(p)}_{(ij)}$ may be derived from (8·4) by differentiation; for we have

$$ \frac{D}{Dt}(ds^2 - ds_0^2) = 2^{(1)}\alpha_{ij}\, d^{(1)}\theta^i\, d^{(1)}\theta^j = 2^{(1)}\alpha_{(ij)}\, ds_i\, ds_j, $$

$$ \frac{D^p}{Dt^p}(ds^2 - ds_0^2) = 2^{(1)}\alpha^{(p)}_{ij}\, d^{(1)}\theta^i\, d^{(1)}\theta^j = 2^{(1)}\alpha^{(p)}_{(ij)}\, ds_i\, ds_j, $$

giving
$$\left.\begin{aligned} {}^{(1)}\alpha_{(ij)} &= \frac{\mathrm{D}^{(1)}\eta_{(ij)}}{\mathrm{D}t} = \frac{{}^{(1)}\alpha_{ij}}{\surd({}^{(1)}\gamma_{ii}{}^{(1)}\gamma_{jj})}, \\ {}^{(1)}\alpha^{(p)}_{(ij)} &= \frac{\mathrm{D}^p({}^{(1)}\eta_{(ij)})}{\mathrm{D}t} = \frac{{}^{(1)}\alpha^{(p)}_{ij}}{\surd({}^{(1)}\gamma_{ii}{}^{(1)}\gamma_{jj})}. \end{aligned}\right\} \tag{8·6}$$

By analogy with the elastic case, we may therefore define a curvilinearly aeolotropic material in which the mechanical properties depend upon the strain and its first n time derivatives by the relation
$${}^{(1)}\tau_{(rs)} = \tfrac{1}{2}\{f_{(rs)}+f_{(sr)}\}, \tag{8·7}$$
where
$$f_{(rs)} \equiv f_{(rs)}({}^{(1)}\eta_{(ij)}, {}^{(1)}\alpha_{(kl)}, {}^{(1)}\alpha^{(p)}_{(tu)}) \quad (p = 2, 3, \ldots, n) \tag{8·8}$$
are functions of the physical components indicated, which are such that
$$f_{(rs)}/\surd({}^{(1)}\gamma_{rr}{}^{(1)}\gamma_{ss})$$
transform as the components of a contravariant tensor with respect to systems of convected co-ordinates. We then have
$${}^{(1)}\tau^{rs} = \tfrac{1}{2}\{f_{(rs)}+f_{(sr)}\}/\surd({}^{(1)}\gamma_{rr}{}^{(1)}\gamma_{ss}), \tag{8·9}$$
and by a tensor transformation to the convected system θ^i,
$$\tau^{ij} = \frac{\partial\theta^i}{\partial^{(1)}\theta^r}\frac{\partial\theta^j}{\partial^{(1)}\theta^s}{}^{(1)}\tau^{rs} = C^{ij}_{(rs)}f_{(rs)}, \tag{8·10}$$
where
$$C^{ij}_{(rs)} = \tfrac{1}{2}\{A^{ij}_{(rs)}+A^{ji}_{(rs)}\} \quad \text{and} \quad A^{ij}_{(rs)} = \frac{\partial\theta^i}{\partial^{(1)}\theta^r}\frac{\partial\theta^j}{\partial^{(1)}\theta^s}\Big/\surd({}^{(1)}\gamma_{rr}{}^{(1)}\gamma_{ss}). \tag{8·11}$$

Symmetry properties may be introduced by analogy with the rectilinear case. Thus a material is defined as being orthotropic with respect to the curvilinear system ${}^{(1)}\theta^i$ if its mechanical properties at any point, referred to the ${}^{(1)}\theta^i$-curves (or the $\bar{\theta}^i$-curves) in the undeformed body at rest, are symmetrical about the tangent planes to the surfaces ${}^{(1)}\theta^i =$ constant through that point. This implies that the quantities ${}^{(1)}\tau_{(rs)}$ when expressed as functions of the physical components ${}^{(1)}\eta_{(rs)}$, ${}^{(1)}\alpha_{(rs)}$, ${}^{(1)}\alpha^{(p)}_{(rs)}$ remain invariant in form under all transformations of co-ordinates in the undeformed body of the type
$$({}^{(1)}\theta^1, {}^{(1)}\theta^2, {}^{(1)}\theta^3) = (\pm\theta^{*1}, \pm\theta^{*2}, \pm\theta^{*3}).$$
Similarly, transverse isotropy with respect to the ${}^{(1)}\theta^1$-direction, implies symmetry about the tangent plane and the normal at each point to the surfaces ${}^{(1)}\theta^1 =$ constant in the undeformed body, that is, form invariance of the functions ${}^{(1)}\tau_{(rs)}$ under all transformations
$${}^{(1)}\theta^1 = \pm\theta^{*1}, \quad {}^{(1)}\theta^\alpha = {}^{(1)}\theta^\alpha(\theta^{*2}, \theta^{*3}) \quad (\alpha = 2, 3),$$
of co-ordinates in the initial state.

If we restrict the functions $f_{(rs)}$ to be polynomial functions of the kinematic quantities ${}^{(1)}\eta_{(ij)}$, ${}^{(1)}\alpha_{(ij)}$, the reductions of §§ 4 and 6 may be carried through without modification, apart from the replacement of t^{ij}, e_{ij}, a_{ij} and A^{ij}_{rs} by ${}^{(1)}\tau_{(ij)}$, ${}^{(1)}\eta_{(ij)}$, ${}^{(1)}\alpha_{(ij)}$ and $A^{ij}_{(rs)}$, respectively. In particular, we may observe that the formulae (4·14) to (4·18) and (6·21) require only these modifications. It may, however, be noticed, that by virtue of relations of the type
$${}^{(1)}\eta_{(ii)} = {}^{(1)}\eta^i_i, \quad {}^{(1)}\eta_{(ij)}{}^{(1)}\eta_{(jk)}\ldots{}^{(1)}\eta_{(lm)}{}^{(1)}\eta_{(mi)} = {}^{(1)}\eta^i_j{}^{(1)}\eta^j_k\ldots{}^{(1)}\eta^l_m{}^{(1)}\eta^m_i$$

which apply irrespective of whether or not summation is carried out over repeated indices, the invariants appropriate to the orthotropic case may be derived from (4·14) by replacing e_{ij}, a_{ij} by ${}^{(1)}\eta^i_j$, ${}^{(1)}\alpha^i_j$, respectively. A similar remark applies to the invariants (5·21) for transversely isotropic bodies, and the functions Θ_{tt}, Θ^r_{tt} which occur in (4·17), (4·18) may appropriately be written in the mixed form Θ^t_t, $\Theta^{(r)t}_t$, respectively.

For curvilinearly aeolotropic bodies of the type considered in the present section the dissipation function Φ may be expressed as a function, apart from physical constants, entirely in terms of the kinematic quantities ${}^{(1)}\eta_{(ij)}$, ${}^{(1)}\alpha_{(ij)}$, ${}^{(1)}\alpha_{(ij)}$.

9. Constraints

In the theory of large elastic deformations, the stress-strain relations for materials subject to constraints may be established from variational principles and based on the concept of a strain-energy function. When time derivatives of the strain enter into the constitutive equations it is not immediately obvious how such considerations can be applied, although the procedure of adding an arbitrary isotropic tensor to the stress to account for the existence of incompressibility can be justified without difficulty. A more general discussion based upon the expression for the stress power is possible if we postulate the existence of geometrical constraints, which are such that throughout the motion they merely restrict the possible configurations which are available to the body without themselves doing any work. This implies that any terms introduced into the expressions for the stress components to account for these constraints must be such that they do not affect the rate of working at any instant.

Following the method employed by Ericksen & Rivlin (1954) and by the author (1956) for elastic materials, we assume that the constraints can be expressed by means of functional relations

$$f_m({}^{(2)}\eta_{ij}) = 0 \quad (m = 1, 2, 3, \ldots) \tag{9·1}$$

between the components of strain ${}^{(2)}\eta_{ij}$ referred to a convected co-ordinate system ${}^{(2)}\theta^i$ which may, or may not coincide with either of the co-ordinate systems θ^i, ${}^{(1)}\theta^i$ previously employed. As in the theory of elasticity, the existence of six independent constraints, i.e. six functionally independent relations between the six quantities ${}^{(2)}\eta_{ij}$ would imply that these could only exist for isolated values of their arguments, and this would limit the motions to those which characterize a rigid body. We therefore impose the condition $m<6$. By differentiating (9·1) we obtain the conditions

$$\frac{\mathrm{D}f_m}{\mathrm{D}t} = \frac{\partial f_m}{\partial {}^{(2)}\eta_{ij}} \frac{\mathrm{D}{}^{(2)}\eta_{ij}}{\mathrm{D}t} = \frac{\partial f_m}{\partial {}^{(2)}\eta_{ij}} {}^{(2)}\alpha_{ij} = 0 \quad (m<6),$$

or alternatively

$$\frac{\partial f_m}{\partial {}^{(2)}\eta_{ij}} \frac{\partial \theta^r}{\partial {}^{(2)}\theta^i} \frac{\partial \theta^s}{\partial {}^{(2)}\theta^j} \alpha_{rs} = 0 \quad (m<6). \tag{9·2}$$

Let τ^{ij}_e denote an expression of the type (4·18) or (6·21), or any of the analogous formulae which may be derived by the procedures of §§ 7 and 8, appropriate to a material for which constraints are absent. The stress power, or dissipation function Φ, which represents the rate at which internal mechanical work is being done per unit volume of the configuration at the current time t, may then be written as

$$\Phi = \tau^{rs}_e \alpha_{rs}.$$

If, owing to the constraint represented by $f_1({}^{(2)}\eta_{rs}) = 0$ the stresses are increased to the extent of an arbitrary symmetric tensor p_1^{rs} we have

$$\Phi = \{\tau_e^{rs} + p_1^{rs}\}\alpha_{rs} = \tau_e^{rs}\alpha_{rs},$$

or

$$p_1^{rs}\alpha_{rs} = 0, \tag{9·3}$$

by virtue of the assumption that the stress power is to remain unaltered. Remembering the symmetry property of α_{rs} and p_1^{rs} we may derive from (9·2) and (9·3) the relation

$$\left\{p_1^{rs} - \tfrac{1}{2}q_1\left(\frac{\partial\theta^r}{\partial^{(2)}\theta^i}\frac{\partial\theta^s}{\partial^{(2)}\theta^j} + \frac{\partial\theta^r}{\partial^{(2)}\theta^j}\frac{\partial\theta^s}{\partial^{(2)}\theta^i}\right)\frac{\partial f}{\partial^{(2)}\eta_{ij}}\right\}\alpha_{rs} = 0, \tag{9·4}$$

q_1 being an arbitrary scalar function of θ^i and t. It is evident that p_1^{rs} must be such that by a suitable choice of q_1 all of the coefficients of α_{rs} may be made to vanish simultaneously. For, if not, it would be possible to choose q_1 so that (9·4) contains non-zero terms, but has at least one vanishing coefficient which corresponds to a non-zero coefficient of (9·2). This would imply an independent linear relation between the components α_{rs} additional to that derived from the constraint condition $f_1({}^{(2)}\eta_{rs}) = 0$. We may therefore take

$$p_1^{rs} = \tfrac{1}{2}q_1\left(\frac{\partial\theta^r}{\partial^{(2)}\theta^i}\frac{\partial\theta^s}{\partial^{(2)}\theta^j} + \frac{\partial\theta^r}{\partial^{(2)}\theta^j}\frac{\partial\theta^s}{\partial^{(2)}\theta^i}\right)\frac{\partial f_1}{\partial^{(2)}\eta_{ij}}.$$

A similar argument applied to each of the constraint conditions (9·2) yields, for the stress tensor, the formula

$${}^{(1)}\tau^{rs} = \tau_e^{rs} + \tfrac{1}{2}\sum_{m=1}^{n} q_m\left(\frac{\partial\theta^r}{\partial^{(2)}\theta^i}\frac{\partial\theta^s}{\partial^{(2)}\theta^j} + \frac{\partial\theta^r}{\partial^{(2)}\theta^j}\frac{\partial\theta^s}{\partial^{(2)}\theta^i}\right)\frac{\partial f_m}{\partial^{(2)}\eta_{ij}} \quad (n<6), \tag{9·5}$$

Incompressible materials provide an obvious illustration of this result. For if there are no volume changes during the motion, we have at each instant the condition

$$|\,\delta_s^r + 2{}^{(2)}\eta_s^r\,| = |\,\delta_s^r + 2\eta_s^r\,| = 1,$$

which, with (9·5) gives

$$\tau^{ij} = \tau_e^{ij} + q_1\Gamma^{ij}, \tag{9·6}$$

consistent with the theory of elasticity and the relations employed by other workers.

If, throughout the motion, lines following the ${}^{(2)}\theta^i$ co-ordinate curves remain unchanged in length we have the single constraint condition

$$f_1({}^{(2)}\eta_{ij}) \equiv {}^{(2)}\eta_{ii} = 0 \quad (i \text{ not summed}),$$

and this, with (9·5) yields

$$\tau^{rs} = \tau_e^{rs} + q_1\frac{\partial\theta^r}{\partial^{(2)}\theta^i}\frac{\partial\theta^s}{\partial^{(2)}\theta^i} \quad (i \text{ not summed}). \tag{9·7}$$

Such conditions may be simulated by the introduction of a system of thin, flexible, inextensible cords along the ${}^{(2)}\theta^i$-curves, and a direct calculation of the stress-deformation relations for a composite material of this kind, along the lines followed by the author (1956) for reinforced elastic sheets, serves to verify (9·7).

In place of (9·1) it is evidently possible to postulate the existence of non-integrable relations of the form

$$F_m^{ij\,(2)}\alpha_{ij} = 0 \quad (F_m^{ij} = F_m^{ji}),$$

in which F^i_m are arbitrary tensor functions of the strain components ${}^{(2)}\eta_{rs}$. The resulting analysis, leading to the equation

$$\tau^{rs} = \tau^{rs}_e + \tfrac{1}{2}\sum_{m=1}^{n} q_m \left(\frac{\partial\theta^r}{\partial^{(2)}\theta^i}\frac{\partial\theta^s}{\partial^{(2)}\theta^j} + \frac{\partial\theta^r}{\partial^{(2)}\theta^j}\frac{\partial\theta^s}{\partial^{(2)}\theta^i}\right) F^{ij}_m,$$

may be compared with the theory of non-holonomic constraints in classical rigid dynamics, but its physical significance in the present instance is not immediately obvious.

10. Transformation to fixed co-ordinate system

The preceding analysis has been concerned with materials for which the strain tensor enters explicitly into the constitutive equations and for which there is therefore a preferred initial configuration. The situation in a given element of the body during its motion is then often of primary interest rather than that at a fixed point in space. For this reason, and also to simplify the analysis, convected co-ordinate systems have been used throughout. If, however, it is required to refer the motion to a co-ordinate system y^i which is fixed in space, the appropriate fixed components of the kinematic and mechanical tensors must be employed. The formation of such components has been discussed by Oldroyd (1950), who has shown that if an unweighted tensor $\beta^{\cdot\,l\,\cdots}_{\cdot j\,\cdots}$ has fixed components $b^{\cdot\,i\,\cdots}_{\cdot k\,\cdots}$, the tensor whose convected components are $\mathrm{D}\beta^{\cdot\,l\,\cdots}_{\cdot j\,\cdots}/\mathrm{D}t$ has fixed components

$$\tilde{b}^{\cdot\,i\,\cdots}_{\cdot k\,\cdots} \equiv \frac{\partial b^{\cdot\,i\,\cdots}_{\cdot k\,\cdots}}{\partial t} + v^m b^{\cdot\,i\,\cdots}_{\cdot k\,\cdots,\,m} + \Sigma v^m_{,k} b^{\cdot\,i\,\cdots}_{\cdot m\,\cdots} - \Sigma' v^i_{,m} b^{\cdot\,m\,\cdots}_{\cdot k\,\cdots}, \qquad (10\cdot1)$$

where $\partial/\partial t$ denotes partial differentiation with respect to t holding the fixed co-ordinates y^i constant and v^i is the contravariant velocity vector in the co-ordinate system y^i. The comma now signifies covariant differentiation with respect to the co-ordinates y^i and the fixed components $G_{ij}(y^r)$, $G^{ij}(y^r)$ of the metric tensors $\Gamma_{ij}(\theta^r, t)$, $\Gamma^{ij}(\theta^r, t)$ and $\Sigma(\Sigma')$ denotes a sum of all similar terms, one for each covariant (contravariant) suffix.

To derive expressions for the stress components referred to the fixed curvilinear reference frame y^i, we choose this system so that at the instant t under consideration the convected system θ^i coincides with it. We denote by $T^{ij}, g_{ij}, g^{ij}, G_{ij}, G^{ij}, E_{ij}, A_{ij}, A^{(r)}_{ij}$ the fixed components, referred to the co-ordinates y^i, of the tensors τ^{ij}, γ_{ij}, γ^{ij}, Γ_{ij}, Γ^{ij}, η_{ij}, α_{ij}, $\alpha^{(r)}_{ij}$ respectively. The metric tensor components $\Gamma_{ij}(\theta^r, t)$, $\Gamma^{ij}(\theta^r, t)$ are therefore replaced, in the system y^i, by the fixed components $G_{ij}(y^r)$, $G^{ij}(y^r)$ which, with the present choice of co-ordinate systems, are equal to them; a similar remark applies to the tensors g_{ij}, g^{ij}, T^{ij} and E_{ij}. From (10·1), we obtain, for the kinematic tensors

$$\left.\begin{aligned} E_{ij} &= \tfrac{1}{2}(G_{ij} - g_{ij}), \\ A_{ij} &= \tfrac{1}{2}(v_{i,j} + v_{j,i}), \\ A^{(r+1)}_{ij} &= \frac{\partial A^{(r)}_{ij}}{\partial t} + v^m A^{(r)}_{ij,\,m} + v^m_{,i} A^{(r)}_{mj} + v^m_{,j} A^{(r)}_{im} \quad (r = 1, 2, \ldots;\ A^{(1)}_{ij} = A_{ij}). \end{aligned}\right\} \qquad (10\cdot2)$$

A similar procedure may be employed to derive fixed components of the mechanical and kinematic tensors ${}^{(r)}\tau^{ij}$, ${}^{(r)}\gamma_{ij}$, ${}^{(r)}\gamma^{ij}$, ... referred to the other convected co-ordinate systems ${}^{(r)}\theta^i$. In each case, the corresponding fixed system ${}^{(r)}y^i$ is chosen to coincide, at current time

t, with the convected system under consideration. Alternatively, these components may be derived by tensor transformations. For example

$$^{(r)}A_{ij} = \frac{\partial y^m}{\partial^{(r)}y^i}\frac{\partial y^n}{\partial^{(r)}y^j}A_{mn}.$$

Corresponding to the physical quantities $^{(1)}\eta_{(ij)}$, $^{(1)}\alpha^{(r)}_{(ij)}$ defined by (8·2) and (8·6) we have

$$^{(1)}E_{(ij)} = \frac{^{(1)}E_{ij}}{\sqrt{(^{(1)}g_{ii}\,^{(1)}g_{jj})}},$$

and

$$^{(1)}A^{(r)}_{(ij)} = \frac{^{(1)}A^r_{ij}}{\sqrt{(^{(1)}g_{ii}\,^{(1)}g_{jj})}},$$

respectively.

To refer the constitutive equations to the fixed reference frame y^i all kinematic and mechanical tensors which occur in the corresponding equation referred to the convected system θ^i are replaced by the appropriate fixed components. In addition, the quantities A^{ij}_{rs}, C^{ij}_{rs}, $A^{ij}_{(rs)}$, $C^{ij}_{(rs)}$ defined in §§4, 5 and 8 are replaced by the quantities

$$^{(f)}A^{ij}_{rs} = \frac{\partial y^i}{\partial Y^r}\frac{\partial y^j}{\partial Y^s}, \quad ^{(f)}C^{ij}_{rs} = \tfrac{1}{2}\{^{(f)}A^{ij}_{rs} + {}^{(f)}A^{ij}_{sr}\},$$

and

$$^{(f)}A^{ij}_{(rs)} = \frac{\partial y^i}{\partial^{(1)}y^r}\frac{\partial y^j}{\partial^{(1)}y^s}\Big/\sqrt{(^{(1)}g_{rr}\,^{(1)}g_{ss})}, \quad ^{(f)}C^{ij}_{(rs)} = \tfrac{1}{2}\{^{(f)}A^{ij}_{(rs)} + {}^{(f)}A^{ji}_{(rs)}\},$$

respectively, Y^i being the fixed co-ordinate system chosen so that at the current time t the convected system x^i coincides with it.

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